Grade 11/12 Math Circles
November 23, 2022
Generating Functions

What can Generating Functions do?

Today we are talking about one of the fundamental tools in a field of mathematics known as Combinatorics: generating functions. To begin, let's look at some problems which we will be able to solve with generating functions!

The Coin Problem

How many ways can we make x cents using 5, 10 and 25 cent coins?

Let's start by considering a small example.

Exercise 1

How many ways can we make 50 cents using just 5, 10 and 25 cent coins?

Stop and Think

What approach did you take to solve Exercise 1? What if we wanted to know the number of ways to make \$1? Or \$5? Does your approach scale to larger amounts of money?

Sicherman Dice

Find two six-sided dice such that:

- *Each side has a positive integer number of dots*
- *The two dice are not the same*
- *The probabilities of rolling a sum of $2, \ldots, 12$ on these dice is the same as the probabilities for regular six-sided dice*

Before we can think about this problem, we first need to understand what it means for the probabilities that we roll $2, \ldots, 12$ to be the same.

With a normal six-sided die, we have the faces $1, 2, 3, 4, 5, 6$. Rolling two such dice gives us 36 combinations:

	1	**2**	**3**	**4**	**5**	**6**
1						
2						
3						
4						
5						
6						

There is only one way, rolling a 1 and then another 1, that we can roll the dice to get a sum of 2. There are two ways, rolling a 1 then 2 or a 2 then 1, to get a sum of 3.

Exercise 2

How many ways can we roll the dice to get a sum of 9? What are those ways?

Stop and Think

Is there a pattern to the number of ways to roll the dice to get a given sum?

So, having the **same probabilities** is having the same number of ways that we can roll each value.

Generating Functions

In order to understand generating functions, we first need a few definitions.

Definition 1. A **set** is a collection of objects. (If there are duplicates, then we assign each of them a different colour.)

Example: Sets

Each of the following are examples of **sets**:

- $\{a, b, c\}$
- $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$
- Set of 0-1 strings $= \{\varnothing, 0, 1, 00, 10, 01, 11, 000, \ldots\}$

Definition 2. A **weight function** is a mapping from a set to $\mathbb{N} = \{0, 1, 2, \ldots\}$.

Example: $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$

Define our weight function w, as $w(a) = 1$, $w(b) = 5$ and $w(c) = 2$.

Example: 0-1 Strings

Define our weight function, w as the length of the string. For example, $w(010) = 3$.

Definition 3. A **combinatorial class** is a set, paired with a weight function, such that there are a finite number of objects of any weight.

Example: Combinatorial Class

The set 0-1 strings with the weight function $w(x) =$ length of the string is a combinatorial class.

Stop and Think

0-1 strings with a weight function of $g(x) =$ the number of 0's in the string is not a combinatorial class. Why not?

Definition 4. A **generating function** is a structure of the form

$$\sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \ldots$$

which stores information about combinatorial classes. In particular, $a_n =$ the number of objects of weight n.

Exercise 3

What is the generating function for the combinatorial class with a set of length three 0-1 strings and weight function $w(x)$ = the number of 1's in the string?

Example: 0-1 String Generating Function

What is the generating function for the combinatorial class with the set of 0-1 strings and the weight function $w(x)$ = length of the string?

By listing out the options, we know that there is 1 string of length zero, 2 of length one, 4 of length two and 8 of length three. In fact, this doubling pattern continues.

This is because each string of length $n+1$ is actually a string of length n with a 0 or a 1 in front. Try creating the length three 0-1 strings from the length two 0-1 strings by putting a 0 or a 1 in front of each string to justify this for yourself:

This gives a generating function of:

$$\begin{aligned} & 1z^0 + 2z^1 + 4z^2 + 8z^3 + \dots \\ = \; & 2^0z^0 + 2^1z^1 + 2^2z^2 + 2^3z^3 + \dots \\ = \; & \sum_{n=0}^{\infty} 2^n z^n \end{aligned}$$

Notation: If $F(z) = \sum_{n=0}^{\infty} a_n z^n$, then the coefficient of z^n is denoted by $[z^n]F(z) = a_n$.

Example: Coefficients

Let $F(z) = 1 + 5z + 3z^2 + 2z^3$. Then, $[z^1]F(z) = 5$.

Generating Function Operations

Throughout this section, let $\mathcal{A}$ and $\mathcal{B}$ be combinatorial classes with corresponding weight functions w_A and w_B. Let the generating functions for $\mathcal{A}$ and $\mathcal{B}$ be $A(z) = \sum_{n=0}^{\infty} a_n z^n$ and $B(z) = \sum_{n=0}^{\infty} b_n z^n$.

Geometric Series

Geometric series give us a way to write our generating functions as rational functions (functions with a polynomial in the numerator and denominator) rather than sums. Let $F(z) = 1 + z + z^2 + z^3 + \ldots$. Then, $F(z) = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$.

Example: Geometric Series 1

$$\begin{aligned} F(z) &= 1 + 2z + 4z^2 + 8z^3 + \ldots \\ &= \sum_{n=0}^{\infty} (2z)^n \\ &= \frac{1}{1-2z} \end{aligned}$$

Example: Geometric Series 2

$$\begin{aligned} H(z) &= z + z^2 + z^3 + z^4 + \ldots \\ &= z(1 + z + z^2 + z^3 + \ldots) \\ &= z\sum_{n=0}^{\infty} z^n \\ &= \frac{z}{1-z} \end{aligned}$$

Addition

$\mathcal{A} + \mathcal{B}$ is defined to give the following combinatorial class: the **set** is the objects in the set of $\mathcal{A}$ as well as the objects in the set of $\mathcal{B}$. The **weight function** gives the weight of the object from its original class. The **generating function** $A(z) + B(z) = \sum_{n=0}^{\infty} (a_n + b_n) z^n$.

Example: Addition

How many 0-1 strings of length two or three have one 1?

In this case, define $\mathcal{A}$ and $\mathcal{B}$ as the following:

$\mathcal{A}$: **Set:** 0-1 strings of length two
Weight Function: The number of 1's
GF: $A(z) = 1 + 2z + z^2$

$\mathcal{B}$: **Set:** 0-1 strings of length three
Weight Function: The number of 1's
GF: $B(z) = 1 + 3z + 3z^2 + z^3$

This gives:

$\mathcal{A} + \mathcal{B}$: **Set:** 0-1 strings of length two or three
Weight Function: The number of 1's
Generating Function: $A(z) + B(z) = 1 + 2z + z^2 + 1 + 3z + 3z^2 + z^3 = 2 + 5z + 4z^4 + z^3$

So, the number of 0-1 strings of length two or three which have one 1 is $[z](A(z) + B(z)) = 5$.

Multiplication

$\mathcal{A} * \mathcal{B}$ is defined to give the following combinatorial class: the **set** is the objects of the form (a, b) where a is in the set of $\mathcal{A}$ and b is in the set of $\mathcal{B}$. The **weight function** is $w((a,b)) = w_A(a) + w_B(b)$. The **generating function** $A(z)B(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) z^n$. Note that the generating function is exactly as we would expect if we were simply multiplying polynomials.

Example: Multiplication 1

What is the generating function for the combinatorial class representing rolling two regular 6-sided dice?

In this case, define $\mathcal{A}$ and $\mathcal{B}$ as the following:

$\mathcal{A} = \mathcal{B}$:

Set: Ways to roll $\{1, 2, 3, 4, 5, 6\}$

Weight Function: Value shown on the die: $w_A(x) = w_B(x) = x$

GF: $A(z) = B(z) = z + z^2 + z^3 + z^4 + z^5 + z^6$

This gives:

$\mathcal{A} * \mathcal{B}$:

Set: (a, b), where a is the first roll and b is the second

Weight Function: $w((a, b)) = a + b$

Generating Function: $A(z)B(z) = (z + z^2 + z^3 + z^4 + z^5 + z^6)^2$

Example: Multiplication 2

Find the generating function for the 0-1 strings ending in 010 *with a weight function of the length of the string.*

Define $\mathcal{A}$ and $\mathcal{B}$ as the following:

$\mathcal{A}$: **Set:** 0-1 strings

Weight Function: Length of string

GF: $A(z) = \frac{1}{1-2z}$

$\mathcal{B}$: **Set:** $\{010\}$

Weight Function: Length of string

GF: $B(z) = z^3$

This gives:

$\mathcal{A} * \mathcal{B}$: **Set:** $(a, 010)$, where a is a 0-1 string

Weight Function: $w((a, 010)) = w_A(a) + 3$

Generating Function: $A(z)B(z) = \frac{z^3}{1-2z}$

Why is $A(z)B(z)$ the generating function that we would like?

Stop and Think

What if we substitute $z = 1$ into a generating function? For example, if $A(z) = 1+3z+3z^2+z^3$, the generating function for 0-1 strings of length 3 with weight function the number of 1's, then $A(1) = 1 + 3 + 3 + 1 = 8$.

Applications of Generating Functions

Now that we have studied generating functions and their properties, we can return to our initial questions, the Coin Problem and Sicherman Dice.

The Coin Problem

How many ways can we make x cents using 5, 10 and 25 cent coins?

To answer this question, we will create three combinatorial classes, one for each of the coin options, representing choosing how many coins of that type to take. Since we want the sum of the coin values to be a certain amount, we will use weight functions representing the value of the coins chosen. For this, we get:

$\mathcal{A}$:
Set:
Choices for number of 5¢ coins
WF: $w_A(x) = 5x$
GF:

$$\begin{aligned} A(z) &= z^0 + z^5 + z^{10} + \dots \\ &= \sum_{n=0}^{\infty} z^{5n} \\ &= \frac{1}{1 - z^5} \end{aligned}$$

$\mathcal{B}$:
Set:
Choices for number of 10¢ coins
WF: $w_B(x) = 10x$
GF:

$$\begin{aligned} B(z) &= z^0 + z^{10} + z^{20} + \dots \\ &= \sum_{n=0}^{\infty} z^{10n} \\ &= \frac{1}{1 - z^{10}} \end{aligned}$$

$\mathcal{C}$:
Set:
Choices for number of 25¢ coins
WF: $w_C(x) = 25x$
GF:

$$\begin{aligned} C(z) &= z^0 + z^{25} + z^{50} + \dots \\ &= \sum_{n=0}^{\infty} z^{25n} \\ &= \frac{1}{1 - z^{25}} \end{aligned}$$

Then,

$\mathcal{A} * \mathcal{B} * \mathcal{C}$:

Set: (a, b, c), where a is from $\mathcal{A}$, b is from $\mathcal{B}$ and c is from $\mathcal{C}$. In other words, each element represents a choice of coins.

Weight Function: $w((a, b, c)) = w_A(a) + w_B(b) + w_C(c) = 5a + 10b + 25c$, in other words, the total value of the coins chosen.

Generating Function: $A(z)B(z)C(z) = \frac{1}{(1-z^5)(1-z^{10})(1-z^{25})}$

Sicherman Dice

Find two six-sided dice such that:

- *Each side has a positive integer number of dots*
- *The two dice are not the same*
- *The probability of rolling a sum of* $1, 2, \ldots, 12$ *on these dice is the same as the probabilities if they were regular six-sided dice*

Let's start by converting each of these requirements into generating function terminology. In particular, we want to represent the two new dice rolls as a generating functions.

1. Having a positive integer number of dots means that the new dice cannot have a constant (z^0) term.
2. To have different dice, the corresponding generating functions for each die must be different.
3. To have the same probabilities, the generating function created by multiplying the two dice generating functions together must be the same as for regular dice.
4. To be six-sided, $A(1) = B(1) = 6$.

From earlier, we found that the generating function of multiplying two regular dice is

$$F(z) = (z + z^2 + z^3 + z^4 + z^5 + z^6)^2$$

If we factor this, we get $F(z) = z^2(z+1)^2(z^2+z+1)^2(z^2-z+1)^2$.

In order to satisfy Requirement 1, we need each die to have a z factor. Since we need the dice to be 6-sided and $A(z) = C(z)D(z) \implies A(1) = C(1)D(1)$, we need a factor of $z+1$ and z^2+z+1 for each die. Finally, since the dice cannot be the same, this gives new dice with generating functions

$$A(z) = z(z+1)(z^2+z+1) = z + 2z^2 + 2z^3 + z^4$$

and

$$B(z) = z(z+1)(z^2+z+1)(z^2-z+1)^2 = z + z^2 + z^4 + z^5 + z^6 + z^8$$

So, our two new dice have sides $\{1, 2, 2, 3, 3, 4\}$ and $\{1, 3, 4, 5, 6, 8\}$.